

## SOME REMARKS ON FOLIATIONS WITH MINIMAL LEAVES

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Let  $\mathcal{F}$  be a foliation on a manifold  $X$  of dimension  $n = p + q$ , the leaves being submanifolds of dimension  $p$  and codimension  $q$ . Everything will be assumed to be of class  $C^\infty$ . The question of the existence of a riemannian metric on  $X$  for which all the leaves are minimal submanifolds has been discussed by Rummeler [5] and Sullivan [7]. Assume for simplicity that the tangent bundle  $T\mathcal{F}$  of the leaves of the foliation is orientable and oriented. They prove the following criterion.

**Theorem (Rummeler-Sullivan.)** *Let  $g_0$  be a smooth scalar product on  $T\mathcal{F}$ . It is induced by a riemannian metric  $g$  on  $X$  for which the leaves are minimal submanifolds iff the volume  $p$ -form  $\omega_0$  on the leaves defined by  $g_0$  (and the orientation) is the restriction to the leaves of a  $p$ -form  $\omega$  on  $X$  which is relatively closed, namely,  $d\omega(\xi_1, \dots, \xi_{p+1}) = 0$  if the first  $p$  vector fields  $\xi_i$  are tangent to the leaves.*

Using the above criterion Rummeler and Sullivan proved the existence or the nonexistence of such a metric in many interesting cases. Our goal is to prove that for a compact  $X$  the above condition depends only on the transverse structure of  $\mathcal{F}$ , and to deduce from this some consequences.

We first give a short proof of the Rummeler-Sullivan criterion. Let  $\nu$  be a vector field on a small open set  $U$  of  $X$  such that the local flow  $\varphi_t$  generated by  $\nu$  maps leaves to leaves. Let  $K_0 \subset U$  be a piece of a leaf and let  $K_t = \varphi_t(K_0)$ . Consider a  $p$ -form  $\omega$  on  $X$  extending  $\omega_0$ . Then for  $t = 0$

$$\frac{d}{dt}(\text{volume } K_t) = \frac{d}{dt} \int_{K_t} \omega = \frac{d}{dt} \int_{K_0} \varphi_t^* \omega = \int_{K_0} \nu \cdot \omega = \int_{K_0} di_\nu \omega + i_\nu d\omega,$$

where  $\nu \cdot \omega$  is the Lie derivative of  $\omega$  in the direction of  $\nu$ . Assume there is a metric  $g$  extending  $g_0$  such that all the leaves are minimal. Define  $\omega$  such that  $i_\nu \omega$  vanishes on each leaf for  $\nu$  orthogonal to the leaves. For such a  $\nu$ ,  $(d/dt)(\text{volume } K_t) = 0$ , hence the above formula shows that  $di_\nu \omega$  vanishes on the leaves, so  $\omega$  is relatively closed.

Conversely, let  $\omega$  be a relatively closed form extending  $\omega_0$ . At a point  $x \in X$ , the vectors  $\xi$  such that  $i_\xi \omega = 0$  form a vector subspace  $N_x$  of  $T_x X$  complementary to the tangent space of the leaf through  $x$ . Consider any metric  $g$  extending  $g_0$  and such that  $N_x$  is orthogonal to the tangent space of the leaf at  $x$ , for all  $x \in X$ . Then the above formula shows that for  $\nu$  orthogonal to the leaves, the first variation  $(d/dt)(\text{volume } K_t)$  is zero for any piece  $K_0$  in a leaf. So each leaf is a minimal submanifold.

## 1. FORMS AND CURRENTS ON THE TRANSVERSE STRUCTURE OF A FOLIATION

### 1.1. Morphisms of pseudogroups

Recall that a pseudogroup  $H$  of diffeomorphisms of a manifold  $T$  is a collection of diffeomorphisms of open sets of  $T$  on open sets of  $T$ , which contains the identity map of  $T$  and is closed under composition (whenever it is defined), inverses, restrictions to open sets, and unions.

Consider two pseudogroups  $H$  and  $H'$  of diffeomorphisms of  $T$  and  $T'$  respectively. A morphism  $\Phi : H \rightarrow H'$  is a collection  $\Phi$  of diffeomorphism of open sets of  $T$  on open sets of  $T'$  such that:

- (i) the sources of the  $\varphi \in \Phi$  cover  $T$ ,
- (ii) if  $h \in H$  and  $\varphi_1, \varphi_2 \in \Phi$ , then  $\varphi_1 h \varphi_2^{-1} \in H'$ ,
- (iii) if  $h \in H, h' \in H', \varphi \in \Phi$ , then  $h' \varphi h \in \Phi$ ,
- (iv)  $\Phi$  is closed under unions.

Any collection  $\Phi_0$  such that

- (a) the  $H$ -orbit of each point of  $T$  intersects the source of a  $\varphi \in \Phi$ ,
- (b) if  $h \in H$ , and  $\varphi_1, \varphi_2 \in \Phi_0$ , then  $\varphi_1 h \varphi_2^{-1} \in H'$  can be uniquely completed as a collection  $\Phi$  satisfying (i)–(iv) by considering all unions of elements of the form  $h' \varphi h, \varphi \in \Phi, h \in H, h' \in H'$ . Such a  $\Phi_0$  will be called an atlas generating the morphism  $\Phi$ .

If  $\Phi'$  is a morphism of  $H'$  in  $H''$ , then the collection of all  $\varphi' \cdot \varphi, \varphi \in \Phi, \varphi' \in \Phi'$ , generates a morphism of  $H$  in  $H''$ . Under this composition, morphisms form a category.  $\Phi_0$  generates an isomorphism (or an equivalence) of  $H$  on  $H'$  iff the union of the targets of the  $\varphi \in \Phi_0$  intersects each orbit of  $H'$  and  $\varphi_2^{-1} h' \varphi_1 \in H$  for any  $\varphi_1, \varphi_2 \in \Phi_0, h' \in H$ . In that case we say that  $H$  is equivalent to  $H'$ . For instance, let  $U$  be an open subset of  $T$  and let  $H_U$  be the pseudogroup of diffeomorphisms of  $U$  whose elements are the restriction to  $U$  of the elements of  $H$ . Then the inclusion of  $U$  in  $T$  generates a morphism of  $H_U$  in  $H$ , and is an isomorphism iff  $U$  meets each orbit of  $H$ . In the case where the space  $T/H$  of  $H$ -orbits is a differentiable manifold, the

natural projection  $p : T \rightarrow T/H$  being locally a diffeomorphism,  $H$  is equivalent to the trivial pseudogroup on  $T/H$  (generated by the identity).

**1.2. Forms and currents on  $T/H$**

Let  $\Omega_c^p(T)$  be the vector space of smooth  $p$ -forms on  $T$  with compact support, and denote by  $\Omega_c^p(T/H)$  the quotient of  $\Omega_c^p(T)$  by the vector subspace generated by elements of the form  $\alpha - h^*\alpha$ , where  $h \in H$ , and  $\alpha$  is a  $p$ -form with compact support in the range of  $h$ . On  $\Omega_c^p(T/H)$  we consider the topology obtained by taking the quotient of the usual  $C^\infty$ -topology on  $\Omega_c^p(T)$ . In general this topology is not Hausdorff (see examples below).

The exterior differential  $d : \Omega_c^p(T) \rightarrow \Omega_c^{p+1}(T)$  induces a continuous differential

$$d : \Omega_c^p(T/H) \rightarrow \Omega_c^{p+1}(T/H).$$

Thus we associated to  $H$  a differential graded topological vector space  $\Omega_c^*(T/H)$ . We shall see below that it depends only on the equivalence class of  $H$ , and its dual is naturally isomorphic to the space of invariant currents on  $T$ . Indeed let  $C_p(T)$  be the space of  $p$ -currents on  $T$ , namely, the vector space of continuous linear forms on  $\Omega_c^p(T)$ . A  $p$ -current  $c$  is invariant by  $H$  if for any  $h \in H$  and any  $p$ -form  $\alpha$  with support in the range of  $h$ , then  $c(\alpha) = c(h^*\alpha)$ . So it defines a continuous linear form on the quotient  $\Omega_c^p(T/H)$ . If  $\alpha \in \Omega_c^p(T)$  is such that  $c(\alpha) = 0$  for all invariant current  $c$ , the class of  $\alpha$  in  $\Omega_c^p(T/H)$  is not zero in general, but is in the closure of the kernel of the projection  $\Omega_c^p(T) \rightarrow \Omega_c^p(T/H)$ . For this reason, it is in general easier to describe the space  $C^p(T)^H$  of invariant currents.

**Proposition.** *A morphism  $\Phi$  of  $H$  in  $H'$  induces functorially a continuous morphism of differential graded vector spaces*

$$\Phi^* : \Omega_c^*(T/H) \rightarrow \Omega_c^*(T'/H').$$

*Proof.* We can express each  $\alpha \in \Omega_c^p(T)$  as a finite sum

$$\alpha = \sum_{\varphi \in \Phi} \alpha_\varphi,$$

where  $\alpha_\varphi$  is a  $p$ -form with compact support in the source  $U_\varphi$  of  $\varphi \in \Phi$ , and is zero except for a finite number of  $\varphi$ .

The map  $\Phi^*$  associates to the class of  $\alpha$  the class  $\sum_\varphi (\varphi^{-1})^* \alpha_\varphi$  in  $\Omega_c^p(T'/H')$ .

We have to check that this definition is independent of the choice of the decomposition of  $\alpha$  and the choice of  $\alpha$  in its class. For the first part, it is sufficient to note that if  $\lambda_\varphi$  is a partition of unity subordinated to the covering

$U_\varphi$  of  $T$ , then

$$\sum_{\varphi} (\varphi^{-1})^* \alpha_{\varphi} = \sum_{\varphi, \psi \in \Phi} (\varphi^{-1})^* \lambda_{\psi} \alpha_{\varphi}$$

is equivalent to

$$\sum_{\varphi, \psi} (\varphi\psi^{-1})^* (\varphi^{-1})^* \lambda_{\psi} \alpha_{\varphi} = \sum_{\varphi, \psi} (\psi^{-1})^* \lambda_{\psi} \alpha_{\varphi} = \sum_{\psi} (\psi^{-1})^* \lambda_{\psi} \alpha$$

because  $\varphi\psi^{-1} \in H'$ .

Assume now that  $\alpha = \beta - h^* \beta$ , where  $\beta$  has its support in the range of  $h \in H$ . We can express  $\beta$  as a finite sum  $\sum \beta_{\varphi}$ , where the support of  $\beta_{\varphi}$  is in the source of  $\varphi$ , and the support of  $h^* \beta_{\varphi}$  is in the source of some  $\psi \in \Phi$  (for this, it is sufficient to multiply  $\beta$  by a partition of unity subordinated to the covering of the range of  $h$  by the intersections  $U_{\varphi} \cap h(U_{\psi})$ ,  $\varphi, \psi \in \Phi$ ). Then  $\alpha = \sum_{\varphi} \beta_{\varphi} - h^* \beta_{\varphi}$  can be mapped on

$$\sum (\varphi^{-1})^* \beta_{\varphi} - \sum (\psi^{-1})^* h^* \beta_{\varphi} = \sum (\varphi^{-1})^* \beta_{\varphi} - \sum (\psi^{-1})^* h^* \varphi^* (\varphi^{-1})^* \beta_{\varphi},$$

which is equivalent to zero because  $\varphi h \psi^{-1} \in H'$ .

It is straightforward to check that  $\Phi^*$  commutes with  $d$  and is continuous.

**Corollary.** *An isomorphism of  $H$  on  $H'$  induces a topological isomorphism of  $\Omega_c^*(T/H)$  on  $\Omega_c^*(T'/H')$ .*

In particular, if  $H$  acts on  $T$  in a properly discontinuous way, i.e., if the map  $T \rightarrow T/H$  is locally a diffeomorphism, then  $\Omega_c^p(T/H)$  is just isomorphic to the vector space of  $p$ -forms with compact support on  $T/H$ .

Also if each point  $x$  of  $T$  has a neighborhood  $V$  such that the restriction of  $H$  to  $V$  is generated by a finite group of diffeomorphisms of  $V$ , then  $T/H$  is a manifold in the sense of Satake, and  $\Omega_c^p(T)$  is what is usually called the space of differential forms on  $T/H$ .

### 1.3. The holonomy pseudogroup of a foliation

Let  $\mathcal{F}$  be a foliation of codimension  $q$  on a manifold  $X$ . A transversal submanifold  $T$  is a manifold of dimension  $q$  together with an immersion  $t : T \rightarrow X$  which is transversal to the leaves.

Given two points  $x_1, x_2$  in  $T$  such that  $t(x_1)$  and  $t(x_2)$  are in the same leaf  $L$ , then a homotopy class of paths  $\gamma$  joining  $t(x_1)$  to  $t(x_2)$  in  $L$  determines a germ at  $x_1$  of a diffeomorphism  $h$  of a neighborhood of  $x_1$  on a neighborhood of  $x_2$ , called the holonomy defined by the path  $\gamma$ : if  $x$  is close to  $x_1$ , there is a path close to  $\gamma$  and contained in a leaf, joining  $t(x)$  to  $t(hx)$ . *The holonomy pseudogroup induced by  $\mathcal{F}$  on  $T$*  is the pseudogroup whose elements are local diffeomorphisms of  $T$  whose germs at each point are determined in this way.

The transversal submanifold  $t : T \rightarrow X$  will be said to be *complete* if  $t(T)$  cuts every leaf of  $\mathcal{F}$ . If  $t' : T' \rightarrow X$  is another complete transversal submanifold, then the holonomy pseudogroup  $H'$  induced on  $T'$  is canonically equivalent to  $H$ . Indeed the set  $\Phi$  of elements of the holonomy pseudogroup induced on the disjoint union of  $T$  and  $T'$  with source in  $T$  and range in  $T'$  is a morphism of  $H$  in  $H'$ . Hence to each foliation  $\mathcal{F}$  we can associate a well defined equivalence class of pseudogroups, namely, the class of any holonomy pseudogroup  $H$  induced by  $\mathcal{F}$  on a complete transversal submanifold  $T$ . By abuse of language, such an  $H$  will be called *the (transverse) holonomy pseudogroup of  $\mathcal{F}$* .

**Definition.** We shall denote by  $\Omega_c^*(\text{Tr } \mathcal{F})$  the topological differential graded vector space of forms on  $T/H$ , where  $H$  is the holonomy pseudogroup induced on a complete transversal submanifold  $T$ . This definition is independent of the choice of the transversal  $T$ , because if  $H'$  is the holonomy pseudogroup induced on a complete transversal submanifold  $T'$ , then  $\Omega_c^*(T/H)$  is canonically isomorphic to  $\Omega_c^*(T'/H')$ .

A continuous linear form on  $\Omega_c^k(\text{Tr } \mathcal{F})$  will be called an *holonomy invariant  $k$ -current*. In other words, it is a  $k$ -current defined on every transversal submanifold, and is invariant by holonomy. The vector space of invariant  $k$ -currents will be denoted by  $C_k(\text{Tr } \mathcal{F})$ . This is the natural generalization of the concept of holonomy invariant measure (cf. [2]). An invariant 0-current will also be called an invariant distribution.

Let  $f : X' \rightarrow X$  be a differentiable map transverse to  $\mathcal{F}$ , and let  $\mathcal{F}' = f^{-1}(\mathcal{F})$  be the foliation on  $X'$  inverse image by  $f$  of  $\mathcal{F}$ . An immersion  $t : T \rightarrow X'$  is a transversal submanifold to  $\mathcal{F}'$  iff  $f \circ t$  is a transversal submanifold to  $\mathcal{F}$ . One has a well-defined morphism of the holonomy pseudogroup induced by  $\mathcal{F}'$  on  $T$  in the holonomy pseudogroup induced by  $\mathcal{F}$  on  $f \circ T$ , hence a functorial morphism

$$\Omega_c^*(\text{Tr } f^{-1}\mathcal{F}) \rightarrow \Omega_c^*(\text{Tr } \mathcal{F}).$$

A *regular covering* of  $\mathcal{F}$  will be a covering of  $X$  by open sets  $U_i$  such that:

(i) The space of leaves of the foliation  $\mathcal{F}_i$  induced by  $\mathcal{F}$  on  $U_i$  is a  $q$ -manifold  $T_i$ , the natural projection  $f_i : U_i \rightarrow T_i$  being a submersion. The inverse images  $f_i^{-1}(y)$ ,  $y \in T_i$ , are the plaques in  $U_i$ .

(ii) Each plaque  $f_i^{-1}(y_i)$  in  $U_i$  meets at most one plaque  $f_j^{-1}(y_j)$ .

Let  $h_{ji}$  be the diffeomorphism mapping  $y_i$  on  $y_j$ ; it is a diffeomorphism of an open set of  $T_i$  on an open set of  $T_j$ . Let  $T$  be the disjoint union of the  $T_i$ , and let  $H$  be the pseudogroup generated by the  $h_{ij}$ . It is easy to see that it is equivalent to the holonomy pseudogroup of  $\mathcal{F}$ ; it will be called *the holonomy pseudogroup associated to the regular covering  $\{U_i\}$* .

## 2. EXAMPLES

## 2.1. Foliations given by closed 1-forms

Let  $T$  be the circle  $R/Z$ , and let  $H$  be the pseudogroup generated by a rotation  $x \mapsto x + \rho$ , where  $\rho$  is an irrational number. The Lebesgue measure is invariant by  $H$ , and any invariant distribution (or  $o$ -current) is a multiple of this measure. Any invariant 1-current is a multiple of the current defined by integration on  $H$ . Hence  $C_0(T)^H$  and  $C_1(T)^H$  are 1-dimensional.

Suppose that  $\rho$  satisfies a diophantine condition: namely, there are positive numbers  $s$  and  $c$  such that

$$|m\rho + n| \geq \frac{c}{(1 + m^2)^s},$$

for any integers  $m, n \neq (0, 0)$ . Then  $\Omega^0(T/H)$  and  $\Omega^1(T/H)$  are isomorphic to  $R$ . Otherwise,  $\rho$  is called a Liouville number; then  $\Omega^0(T/H)$  and  $\Omega^1(T/H)$  are not Hausdorff, but their quotient by the closure of 0 is still isomorphic to  $R$ .

The proof of these facts is a standard argument using Fourier series expansion. A function  $f$  on  $T$  with Fourier series  $\sum_m f_m e^{2imx}$  is  $C^\infty$  iff for each positive integer  $k$ , there is a constant  $c$  such that

$$|f_m| < \frac{c}{(1 + m^2)^k}.$$

$f$  is 0 in  $\Omega^0(T/H)$  iff there is a  $C^\infty$ -function  $g$  such that  $f(x) = g(x) - g(x + \rho)$ . A necessary condition is  $\int_T f(x) dx = f_0 = 0$ , and the Fourier coefficients  $g_m, m \neq 0$ , are uniquely defined ( $\rho$  is irrational). If  $\rho$  satisfies a diophantine condition,  $g_m$  will be the Fourier coefficients of a  $C^\infty$ -function  $g$ ; if  $\rho$  is a Liouville number, this will not be the case for a general  $f$ .

Let  $\mathcal{F}$  be a foliation given on a compact manifold by a closed 1-form  $\omega$ ; the cohomology class of  $\omega$  defines a homomorphism of  $H_1(X, Z)$  in  $R$  whose image is called the group  $P$  of periods of  $\omega$ . The holonomy pseudogroup is equivalent to the pseudogroup of  $T$  generated by the rotations  $x \rightarrow x + \alpha/\alpha_0$ , where  $\alpha_0$  is a fixed nonzero period and  $\alpha \in P$ . The rank of  $P$  is at least one and is larger than one iff every leaf is dense.

More generally, suppose that  $\mathcal{F}$  is given by  $q$  independent closed 1-forms. They define a homomorphism of  $H_1(X, Z)$  in  $R^q$  whose image  $P$  (the group of periods) is of rank  $q$  over  $R$ . If  $X$  is compact, the holonomy pseudogroup is equivalent to the pseudogroup of transformations of  $R^q$  generated by the translations belonging to  $P$ . For everywhere-dense leaves, this is equivalent to the existence of periods  $\alpha, \beta_1, \dots, \beta_q \in P$  such that  $\beta_1, \dots, \beta_q$  are linearly independent over  $R$ , and  $\alpha = a_1\beta_1 + \dots + a_q\beta_q$ , where the real numbers

$1, a_1, \dots, a_q$  are linearly independent over the rationals  $\mathcal{Q}$ . The space  $C_k(\text{Tr } \mathcal{F})$  of  $k$ -invariant currents is isomorphic to the space of  $(q - k)$ -forms on  $\mathbf{R}^q$  invariant by all translations. The quotient of  $\Omega^k(\text{Tr } \mathcal{F})$  by the closure of 0 is isomorphic to the invariant  $k$ -forms on  $\mathbf{R}^q$ , namely, to the dual of the  $k$ -exterior power  $\Lambda^k \mathbf{R}^q$  of  $\mathbf{R}^q$ . However, if  $(a_1, \dots, a_q)$  satisfies a diophantine condition (cf. for instance Hermann [1]), then  $\Omega^k(\text{Tr } \mathcal{F})$  is actually isomorphic to the dual of  $\Lambda^k \mathbf{R}^q$ .

The previous examples are particular cases of transversely homogeneous foliations. Let  $G/H$  be a homogeneous space, where  $H$  is a closed subgroup of the Lie group  $G$ . We assume that  $G$  acts effectively on  $G/H$  and that  $G/H$  is simply connected. A transversely homogeneous foliation  $\mathcal{F}$  on  $X$  is given by an open covering  $\{U_i\}$  and local submersions  $f_i : U_i \rightarrow G/H$  such that the transition diffeomorphisms  $h_{ij}$  are restrictions of translations of  $G/H$  by elements of  $G$ . To such a foliation is associated a homomorphism

$$\Phi : \pi_1(X, x) \rightarrow G$$

whose image  $\Gamma$  is called the global holonomy group of  $\mathcal{F}$ . On the covering  $\tilde{X}$  of  $X$  corresponding to the kernel of  $\Phi$ , the induced foliation  $\tilde{\mathcal{F}}$  is given by a submersion  $f : \tilde{X} \rightarrow G/H$  which is  $\Gamma$ -equivariant,  $\Gamma$  acting on  $\tilde{X}$  by covering translations (cf. Haefliger, Comment. Math. Helv. **32** (1958) 280–281).

If  $X$  and  $H$  are compact, it is easy to see that  $f$  is a fiber map with connected fibers. Then it follows that the holonomy pseudogroup of  $\mathcal{F}$  is generated by  $\Gamma$  acting on  $G/H$ .

In general, for an homogeneous space  $G/H$  of dimension  $n$  (for which the action of  $G$  preserves an orientation), the  $k$ -currents invariant by  $G$  are given by the  $G$ -invariant  $(n - k)$ -forms on  $G/H$  (such a form  $\alpha$  defines the current  $c$  associating to a  $k$ -form  $\omega$  with compact support on  $G/H$  the number  $\int \alpha \wedge \omega$ ).

If  $X$  and  $H$  are compact,  $\mathcal{F}$  has an everywhere-dense leaf iff  $\Gamma$  is dense in  $G$ . In that case the holonomy invariant currents are precisely the  $G$ -invariant forms on  $G/H$ .

## 2.2. Reeb component

Let  $R$  be the solid torus  $S^1 \times D^2$  with a Reeb foliation such that the infinitesimal holonomy group of  $\partial R$  is nontrivial. Then the holonomy pseudogroup is equivalent to the pseudogroup  $H$  of transformations of  $T = [0, \infty[$  generated by  $h : x \rightarrow \lambda x$ , where  $0 < \lambda < 1$ . Thus  $\Omega_c^0(T/H)$  is isomorphic to the space of  $h$ -invariant  $C^\infty$  functions on  $]0, \infty[$  (which is itself isomorphic to space of  $C^\infty$  functions on the circle). The isomorphism maps the class of

$f \in \Omega_c^0(T)$  on the function on  $]0, \infty[$  given by

$$x \rightarrow \sum_{m=-\infty}^{+\infty} \lambda^m x f'(\lambda^m x).$$

Similarly  $\Omega_c^1(T/H)$  is isomorphic to the space of  $h$ -invariant 1-forms on  $]0, \infty[$ .

### 3. INTEGRATION ALONG THE LEAVES

**3.1. Theorem.**  *$\mathcal{F}$  be a foliation on  $X$  with leaves of dimension  $p$ , and assume that the tangent bundle to the leaves is oriented. Then there is a continuous open surjective linear map*

$$\int_{\mathcal{F}} : \Omega_c^{p+k}(X) \rightarrow \Omega_c^k(\text{Tr } \mathcal{F})$$

which commutes with  $d$ .

*Proof.* The construction is directly inspired by the construction of the Ruelle-Sullivan current associated to an invariant measure [4].

First recall that if  $f : X \rightarrow Y$  is a submersion of a  $(p + q)$ -manifold  $X$  in a  $q$ -manifold  $Y$ , the fibers  $f^{-1}(y)$  being coherently oriented, there is a continuous map

$$\int_{\mathcal{F}} : \Omega_c^{p+k}(X) \rightarrow \Omega_c^k(Y)$$

commuting with  $d$ . If  $\omega$  has its support in a coordinate neighborhood where  $f$  is expressed as the linear projection

$$f(x', \dots, x^p, y', \dots, y^q) = (y', \dots, y^q),$$

$$\omega = \sum_J a_J(x, y) dy^J \wedge dx^i \wedge \dots \wedge dx^p$$

+ terms of degree  $< p$  in the  $x^i$ ,

then

$$\int_f \omega = \left( \sum \int a_J(x, y) dx^1 \dots dx^p \right) dy^J.$$

Let  $\{U_i\}$  be a regular covering of  $X$  for  $\mathcal{F}$ , with projections  $f_i : U_i \rightarrow T_i$ . Let  $T$  be the disjoint union of the  $T_i$ , and  $H$  the induced holonomy pseudogroup generated by the  $h_{ij}$  (cf. §1.3.). Given  $\omega \in \Omega_c^{p+k}(X)$ , we can express it as a finite sum  $\omega = \sum \omega_i$ , where the support of  $\omega_i$  is in  $U_i$ .  $\int_{\mathcal{F}} \omega$  will be defined as the class in  $\Omega_c^k(T/H) = \Omega_c^k(\text{Tr } \mathcal{F})$  of  $\sum \bar{\omega}_i$ , where  $\bar{\omega}_i = \int_{f_i} \omega$ . The class of  $\sum \bar{\omega}_i$  is independent of the decomposition of  $\omega$ . Indeed, if  $\{\lambda_j\}$  is a partition of



unity subordinated to  $\{U_i\}$ , then

$$\sum_i \int_{f_i} \omega_i = \sum_{i,j} \int_{f_i} \lambda_j \omega_i$$

is equivalent to

$$\sum_{i,j} \int_{f_j} \lambda_j \omega_i = \sum_j \int_{f_j} \lambda_j \omega,$$

because if the support of  $\alpha$  is in  $U_i \cap U_j$ , then

$$\int_{f_i} \alpha = h_{ij}^* \int_{f_j} \alpha.$$

It is obvious that this map is continuous and commutes with  $d$ . One easily shows that it is independent of the choice of the regular covering (by passing to common refinements).

**Corollary.** *The transpose of  $\int_{\mathfrak{F}}$  gives a linear map*

$$C_k(\text{Tr } \mathfrak{F}) \rightarrow C_{k+p}(X)$$

*of the space of holonomy invariant  $k$ -currents in the space of  $(p+k)$ -currents on  $X$ . This map commutes with the boundary operator  $\delta$ .*

This is a straightforward generalization of the construction of Ruelle-Sullivan [4] associating to an invariant measure a  $p$ -current on  $X$ .

To see an example of a  $p$ -current on  $X$  arising from a holonomy invariant distribution which is not a measure, consider a Reeb foliation like in Example 2.2. Let  $L$  be a noncompact leaf, and  $\xi$  a vector field along  $L$  invariant by holonomy (i.e., projectable with respect to local projections). Let  $\omega$  be a 2-form on  $X$ , and denote by  $\xi.\omega$  its derivative in the direction of  $\xi$  (restricted to  $L$ ). Then  $\int_L \xi.\omega$  is finite and defines a 2-current on  $X$  which arises from a holonomy invariant distribution of order one.

### 3.2. The kernel of $\int_{\mathfrak{F}}$

Following the terminology of [5], a  $(p+k)$ -form is  $\mathfrak{F}$ -trivial if for any sequence  $\xi_1, \dots, \xi_{p+k}$  of vector fields such that  $p$  of them are tangent to  $\mathfrak{F}$ , then  $\omega(\xi_1, \dots, \xi_{p+k}) = 0$ .

**Theorem.** *The kernel of  $\int_{\mathfrak{F}}$  is the vector subspace generated by  $\mathfrak{F}$ -trivial forms and differential of  $\mathfrak{F}$ -trivial forms.*

*Proof.* We first prove the assertion in the particular case of the foliation given by the natural linear submersion  $f: R^q \times R^p \rightarrow R^q$ , where  $f(x, y) = x$ . Any  $(p+k)$ -form  $\omega$  with compact support can be written as  $\omega = \alpha + \beta$ ,

where  $\beta$  is  $\mathcal{F}$ -trivial, and

$$\alpha = \sum a_I dx^I \wedge dy^1 \wedge \cdots \wedge dy^p,$$

where  $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ ,  $1 \leq i_1 < \cdots < i_k \leq q$ . By assumption, for each  $I$ ,  $\int_{\mathbb{R}^p} a_I(x, y) dy^1 \wedge \cdots \wedge dy^p = 0$ . Hence there are smooth  $(p-1)$ -forms  $\gamma_I$  on  $\mathbb{R}^p$  depending smoothly on the parameter  $x$ , such that  $d\gamma_I = a_I dy^1 \wedge \cdots \wedge dy^p$  (cf. [3] where a smooth homotopy operator is constructed). Let  $\gamma = (-1)^k \sum dx^i \wedge \gamma_I$ . Then  $d\gamma = \alpha + \beta'$ , where  $\beta'$  is  $\mathcal{F}$ -trivial. Hence  $\omega = d\gamma - \beta' + \beta$  where  $\gamma$ ,  $\beta$  and  $\beta'$  are  $\mathcal{F}$ -trivial.

We now consider the general case. To construct  $\int_{\mathcal{F}} \omega$  we use as before a regular covering of  $X$  such that each  $f_i : U_i \rightarrow T_i$  is diffeomorphic to a linear projection as above. If  $\omega$  is  $\mathcal{F}$ -trivial, then using a partition of unity we can express it as a finite sum of  $\mathcal{F}$ -trivial forms  $\omega_i$  with support in  $U_i$ . Thus it is clear that

$$\int_{f_i} \omega_i = 0, \quad \int_{f_i} d\omega_i = 0.$$

Conversely, assume that  $\int_{\mathcal{F}} \omega = 0$ . This means that there are  $k$ -forms  $\beta_{ji}$  with compact support in  $T_i$  such that

$$\sum_i \int_{f_i} \omega_i = \sum_{i,j} h_{ij}^*(\beta_{ji}) - \beta_{ji}.$$

Hence

$$\int_{f_i} \omega_i = \sum_j h_{ji}^*(\beta_{ij}) - \beta_{ji}.$$

Let  $\alpha_{ji}$  be  $(p+k)$ -forms with compact support in  $U_i \cap U_j$  such that

$$\int_{f_i} \alpha_{ji} = \beta_{ji}.$$

Note that

$$\int_{f_i} \alpha_{ji} = h_{ij}^*(\beta_{ji}),$$

hence

$$\int_{f_i} \tilde{\omega}_i = 0,$$

where

$$\tilde{\omega}_i = \omega_i - \sum_j (\alpha_{ij} - \alpha_{ji}).$$

It follows from the particular case that each  $\tilde{\omega}_i$  is the sum of a  $\mathcal{F}$ -trivial form and the differential of a  $\mathcal{F}$ -trivial form. But this is also true for  $\omega$  because  $\omega = \sum \tilde{\omega}_i$ .

### 3.3. Interpretation of $\Omega_c^0(\text{tr } \mathcal{F})$

Let  $\Omega_c^r(\mathcal{F})$  be the vector space of smooth  $r$ -forms along the leaves (namely, the smooth sections of the  $r$ th exterior power of the cotangent bundle of the leaves). The differential  $\Omega_c^r(\mathcal{F}) \rightarrow \Omega_c^{r+1}(\mathcal{F})$  along the leaves will be denoted by  $d_0$ . If we denote by  $X_{\mathcal{F}}$  the set  $X$  which is the union of the leaves of  $\mathcal{F}$  and considered as a manifold of dimension  $p$ , then the identity map  $j : X_{\mathcal{F}} \rightarrow X$  is an immersion.  $\Omega_c^r(\mathcal{F})$  is the image in  $\Omega^r(X_{\mathcal{F}})$  by  $j^*$  of  $\Omega_c^r(X)$ , and  $d_0$  is the restriction to  $\Omega_c^r(\mathcal{F})$  of the differential in  $\Omega^r(X_{\mathcal{F}})$ . Let  $H^r(\mathcal{F})$  be the  $r$ -th cohomology group of  $\Omega_c^*(\mathcal{F})$ . This is (almost by definition) the  $r$ -th cohomology group of  $X$  with value in the sheaf of germs of smooth functions which are constant on the leaves.

**Corollary.**  $\Omega^0(\text{Tr } \mathcal{F})$  is canonically isomorphic to  $H_c^p(\mathcal{F})$ , where  $p = \dim \mathcal{F}$ .

Indeed,  $\Omega_c^p(\mathcal{F})$  is just the quotient of  $\Omega_c^p(X)$  by  $\mathcal{F}$ -trivial forms. Also  $j^* d\Omega_c^{p-1}(X) = d_0\Omega_c^{p-1}(\mathcal{F})$ .

## 4. APPLICATIONS TO FOLIATIONS BY MINIMAL LEAVES

Throughout this section we assume  $X$  to be compact and  $\mathcal{F}$  oriented. The following theorem is a direct consequence of the preceding section.

**4.1. Theorem.** A  $p$ -form  $\omega_0$  along  $\mathcal{F}$  with compact support is the restriction of a relatively closed form  $\omega$  with compact support if  $d \int_{\mathcal{F}} \omega_0 = 0$  in  $\Omega_c^1(\text{Tr } \mathcal{F})$ .

*Proof.* Let  $\tilde{\omega}$  be a  $p$ -form with compact support in  $X$  such that  $\omega_0 = j^*\tilde{\omega}$ . As  $\int_{\mathcal{F}} \omega_0 = \int_{\mathcal{F}} \tilde{\omega}$  and  $d \int_{\mathcal{F}} \tilde{\omega} = \int_{\mathcal{F}} d\tilde{\omega} = 0$ , by §3.2 there is a  $p$ -form  $\alpha \in \Omega_c^p(X)$  which is  $\mathcal{F}$ -trivial (i.e.,  $j^*\alpha = 0$ ) such that  $d\tilde{\omega} - d\alpha$  is  $\mathcal{F}$ -trivial. Then  $\omega = \tilde{\omega} - \alpha$  is relatively closed and  $j^*\omega = \omega_0$ .

**Corollary.** Let  $\mathcal{F}$  be an oriented foliation on a compact manifold  $X$ . Let  $g_0$  be a smooth riemannian metric along the leaves and let  $\omega_0$  be the volume form along the leaves defined by  $g_0$  and the orientation of  $\mathcal{F}$ . Then there is a riemannian metric  $g$  on  $X$  inducing  $g_0$  on the leaves and for which the leaves are minimal submanifolds iff  $d \int_{\mathcal{F}} \omega_0 = 0$ .

This follows from the above theorem and the theorem of Rummier-Sullivan mentioned in the introduction (cf. [5] and [6]).

**Corollary 2** (Rummler [6]). *Suppose that the foliation is a generalized Seifert bundle. Then the metric  $g_0$  along the leaves extends to a riemannian metric  $g$  on  $X$  for which all the leaves are minimal iff the volume of each generic leaf  $L$  is constant.*

**Corollary 3.** *Let  $\mathcal{F}$  be a foliation on a compact manifold  $X$  given by a closed 1-form  $\omega$ , and assume that there are at least two  $Q$ -independent periods of  $\omega$ . Then any riemannian metric on the leaves can be approximated in the  $C^\infty$ -topology by a metric which is the restriction to the leaves of a riemannian metric for which the leaves are minimal. If there are two periods whose ratio satisfies a diophantine condition, then any smooth metric on the leaves is the restriction of a metric on  $X$  for which the leaves are minimal.*

This follows from the considerations in Example 2.1.

**Corollary 4.** *Assume there is no holonomy invariant distribution. Then any riemannian metric  $g_0$  on the leaves is close in the  $C^\infty$ -topology to a metric which is the restriction of a riemannian metric on  $X$  for which the leaves are minimal.*

*Proof.* Let  $\omega_0 \in \Omega^p(\mathcal{F})$  be the volume form of  $g_0$ . In any neighborhood of  $\omega_0$  there is a form  $\bar{\omega}_0$  such that  $\int \bar{\omega}_0 = 0$ , because the map  $\int_{\mathcal{F}}$  is open and, by assumption, 0 is dense in  $\Omega_c^p(\text{Tr } \mathcal{F})$ . Now  $\bar{\omega}_0$  is the volume form of a riemannian metric on the leaves close to  $g_0$ . So we can apply Corollary 1.

**Remark.** More generally, the conclusion of the first part of Corollary 3 is still valid for a transversely  $G/H$ -homogeneous foliation  $\mathcal{F}$  on a compact manifold  $X$  with an everywhere dense leaf, assuming  $G$  compact connected. In that case, it follows from §2.1 that the space of holonomy invariant distributions is isomorphic to  $\mathbf{R}$ . Thus the quotient of  $\Omega^p(\text{Tr } \mathcal{F})$  by the closure of zero is isomorphic to  $\mathbf{R}$ , representative for its element being constant functions on  $G/H$  (which is compact by assumption). Hence, if  $\omega_0$  is a volume form on the leaves, then there is a constant  $c$  such that  $\int \omega_0 - c$  is adherent to zero. So we can replace  $\omega_0$  as above by an arbitrary close form  $\bar{\omega}_0$  such that  $\int \bar{\omega}_0$  is equivalent to the constant  $c$ , and hence has zero differential.

**Corollary 5.** *Let  $g_0$  be a riemannian metric on the leaves of an oriented foliation  $\mathcal{F}$  on a compact manifold  $X$ . A necessary condition for  $g_0$  to be arbitrarily close to the restriction to the leaves of a metric for which the leaves are minimal is that*

$$\left\langle c, d \int_{\mathcal{F}} \omega_0 \right\rangle = 0$$

for each holonomy invariant 1-current  $c$ , where  $\omega_0$  is the volume form on the leaves defined by  $g_0$ . This condition is also sufficient if  $\mathcal{F}$  is transversely oriented and of codimension 1.

*Proof.* The necessity follows from Corollary 1, and the sufficiency is implied by the following assertion.

**Claim.** Let  $H$  be a pseudogroup of orientation-preserving local diffeomorphisms of a 1-dimensional manifold  $T$ . Assume that  $T$  has a finite number of connected components, and let  $f$  be a smooth function with compact support on  $T$  such that  $\langle c, df \rangle = 0$  for each  $H$ -invariant 1-current  $c$ . Then arbitrarily close to  $f$  in the  $C^\infty$ -topology, there is a smooth function  $g$  such that  $dg = 0$  in  $\Omega_c^1(T/H)$ .

To prove this we can assume that  $H$  is irreducible in the following sense: we can order the connected components  $T_i$  of  $T$  so that for each  $i$  there is  $h_i \in H$  with source an open set in  $\cup_{j < i} T_j$  and target in  $T_i$ . By assumption, there is a sequence  $\alpha_n \in \Omega_c^1(T)$  such that  $\alpha_n$  converges to  $df$  in the  $C^\infty$ -topology and  $\alpha_n = 0$  in  $\Omega_c^1(T/H)$ . This implies that  $\int_T \alpha_n = 0$ , because integration on  $T$  gives an invariant current. If the integral of  $\alpha_n$  on each  $T_i$  would be zero, then  $\alpha_n$  would be the differential of a function  $f_n$  with compact support on  $T$ , and the sequence  $f_n$  (modified by suitable constants on the compact components  $T_i$ ) would converge to  $f$ .

To achieve this condition, we argue by descending induction on  $r$ . Assume that  $\int_{T_i} \alpha_n = 0$  for each  $i > r$ . Then one can find a sequence  $\alpha'_n$  such that  $\alpha'_n$  converges to  $df$ ,  $\alpha'_n$  is zero in  $\Omega_c^1(T/H)$  and  $\int_{T_i} \alpha'_n = 0$  for  $i > r - 1$ . Indeed, choose a 1-form  $\gamma$  with compact support in the target of  $h_r$  such that  $\int_{T_r} \gamma = 1$ . Then we define

$$\alpha'_n = \alpha_n - c_n \gamma + h_r(c_n \gamma),$$

where  $c_n = \int_{T_r} \alpha_n$ . Note that  $c_n$  tends to zero because  $\int_{T_r} \alpha_n$  converges to  $\int_{T_r} df = 0$ .

**Remark.** Corollary 5 implies the following. Let  $\mathcal{F}$  be an oriented and transversely oriented foliation of codimension one. Then any metric on the leaves is arbitrarily close to the restriction of a metric on  $X$  for which all the leaves are minimal if and only if  $\partial C_1(\text{Tr } \mathcal{F}) = 0$ , where  $\partial : C_1(\text{Tr } \mathcal{F}) \rightarrow C_0(\text{Tr } \mathcal{F})$  is the dual of  $d$ .

As an example (besides the one given in Corollary 3), assume that the holonomy pseudogroup of  $\mathcal{F}$  is equivalent to the pseudogroup generated by a cocompact subgroup  $\Gamma$  of  $PSL_2(\mathbf{R})$  acting as usual on  $S^1$  identified with the boundary of the Poincaré disk  $D$ . (For instance,  $\mathcal{F}$  might be the Anosov foliation associated to the geodesic flow on a compact riemann surface with constant negative curvature). The only  $\Gamma$ -invariant 1-current on  $S^1$  are the multiple of the current defined by integration on  $S^1$ . Indeed any 1-current  $c$  on  $S^1$  is the restriction to  $S^1$  of a harmonic function  $f$  on  $D$ , and if  $c$  is  $\Gamma$ -invariant, then  $f$  is also  $\Gamma$ -invariant, and hence constant because  $\Gamma \backslash D$  is compact. So any  $\Gamma$ -invariant 1-current has a trivial boundary.

**4.1. Theorem.** *On the compact manifold  $X$  there is a metric such that the leaves of  $\mathcal{F}$  are minimal submanifolds iff for a representative  $H$  of the holonomy pseudogroup acting on a  $q$ -manifold  $T$ , there is a smooth positive function  $f$  with compact support, which is strictly positive on a set intersecting each orbit, and satisfies that  $df = 0$  in  $\Omega_c^1(T/H)$ .*

Before giving the proof of this theorem, we state two corollaries.

**Corollary 1.** *The existence of a riemannian metric for which the leaves are minimal depends only on the holonomy pseudogroup of  $\mathcal{F}$ .*

**Corollary 2.** *If there is a representative  $H$  for the holonomy pseudogroup acting on a compact manifold  $T$ , then there is a metric for which the leaves are minimal.*

Indeed we can choose  $f \equiv 1$ . For instance this is the case if the holonomy pseudogroup is generated by a discrete subgroup of a Lie group acting on a compact manifold. Such an example is given by a foliation defined by  $q$  independent closed 1-forms.

*Proof of the theorem.* First we note that the existence of such an  $f$  is independent of the representative for the holonomy pseudogroup.

More precisely, let  $H'$  be a pseudogroup acting on  $T'$  which is equivalent to  $H$  by an isomorphism  $\Phi : H' \rightarrow H$ . Let  $K'$  be a compact set intersecting each orbit of  $H'$ . Then there is a positive smooth function  $f'$  with compact support equivalent to  $f$  and which is strictly positive on  $K'$ . To see that, we choose a finite number of  $\varphi_i \in \Phi$ ,  $i = 1, \dots, r$ , whose domains  $U_i$  cover  $K'$  such that  $f$  is strictly positive on  $\varphi_i(U_i)$ . One can find a covering of  $K$  by compact sets  $K_i \subset K' \cap U_i$ . Let  $\varphi_j$ ,  $r < j \leq s$  be elements of  $\Phi$  such that the ranges of the  $\varphi_k$ ,  $1 \leq k \leq s$ , cover the support  $S$  of  $f$ . Choose a partition of unity  $\lambda_k$  subordinated to the covering of  $T$  by the ranges of the  $\varphi_k$ ,  $1 \leq k \leq s$  (and also the complement of  $S$ ). We can choose the  $\lambda_i$  strictly positive on the  $\varphi_i(K_i)$  for  $1 \leq i \leq r$ . Then

$$f' = \sum_1^s \varphi_i^*(\lambda_i f)$$

is the desired function.

Let  $\{U_i\}$  be a finite regular covering of  $X$  for  $\mathcal{F}$  with local projections  $f_i: U_i \rightarrow T_i$ . We can assume that the  $f_i$  are diffeomorphic to natural projections  $U_i = T_i \times R^q \rightarrow T_i$ . Let  $\{V_i\}$  be a covering of  $X$  by compact sets  $V_i$  contained in  $U_i$ . In each  $U_i$  we can construct a closed  $p$ -form  $\alpha_i$ , whose restriction to each plaque  $P$  of  $U_i$  has compact support, is strictly positive on  $P \cap V_i$ , and satisfies  $\int_P \alpha_i = 1$ . Let  $H$  be the holonomy pseudogroup induced on  $T = \text{union of } T_i$ . By hypothesis and the preceding considerations, we can find a smooth positive function  $f$  with compact support on  $T$ , which is strictly

positive on each  $K_i = f_i(V_i)$  and satisfies that  $df = 0$  in  $\Omega^1(T/H)$ . Let  $g_i$  be the restriction of  $f$  to  $T_i$ .

Then  $\omega = \sum \omega_i$ , where  $\omega_i = f_i^*(g_i)\alpha_i$  is a  $p$ -form on  $X$ , which is positive on the leaves and whose integral over  $\mathcal{F}$  is equivalent to  $f$ . Then we can apply Corollary 1 of Theorem 4.1.

#### 4.3. Examples of foliations having no riemannian metric for which the leaves are minimal

This will be in particular the case for a foliation  $\mathcal{F}$  having a positive holonomy invariant measure which is the boundary of an invariant 1-current (cf. [7]). Indeed in this case, for any  $p$ -form  $\omega_0$  positive on the leaves, we have

$$c\left(d\int_F \omega_0\right) = \partial c\left(\int_F \omega_0\right) > 0.$$

For instance in the case of codimension 1, let  $R$  be a Reeb component with boundary  $\delta R$ ; a transversal curve entering  $R$  cannot cross the boundary again. The 1-current defined by the integral on positively oriented transversal curves is an invariant current whose boundary is the Dirac measure corresponding to  $\delta R$ .

In the case of the horocycle flow (cf. Sullivan [6]), one has on the transverse submanifold a positive invariant 2-form which is the exterior differential of an invariant 1-form (defining an invariant 1-current). This example can be generalized as follows. Let  $G$  be a semisimple Lie group acting on a manifold  $M$  of dimension  $n$  so that the induced action on the space  $T_0^*M$  of nonzero cotangent vectors is transitive. For instance,  $G$  might be the conformal group  $O(n+1, 1)$  acting on the  $n$ -sphere  $S^n$  or the linear group  $SI(n+1, R)$  acting on  $S^n$  identified to the rays in  $R^{n+1}$ .

On  $T^*M$ , one has the canonical 1-form  $\omega$  which is invariant by the differential of any diffeomorphism of  $M$ , and whose exterior differential  $d\omega$  is the canonical symplectic form. Then  $(d\omega)^n$  is a volume form on  $T_0^*M$ , which is the differential of  $\omega \wedge (d\omega)^{n-1}$ . This form defines a 1-current invariant by the differential of any diffeomorphism, and its boundary is the invariant measure defined by  $(d\omega)^n$ .

Let  $\Gamma$  be a discrete subgroup of  $G$  such that  $\Gamma \backslash G$  is compact. Let  $H$  be the subgroup of  $G$  leaving a given covector fixed. Then the cosets  $gH$  are the leaves of a foliation on  $G$  parametrized by the space  $T_0^*M$ . This foliation is invariant by the left action of  $\Gamma$  on  $G$ . So we get on  $\Gamma \backslash G$  a foliation whose transverse structure is  $T_0^*M$ , the holonomy pseudogroup being generated by  $\Gamma$ .

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